$$H(s) = \frac{1}{1 + \left( \frac{s}{\omega_B} \right)^n}$$

$$\Rightarrow |H(j\omega)| = \frac{1}{1 + \left( \frac{\omega}{\omega_B} \right)^{2n}}$$

**Attenuation:**

$$\alpha(\omega) = -20 \log |H(j\omega)|$$

$$|H(\omega)| = 10^{-\frac{\alpha(\omega)}{20}}$$

Define passband corner at $\omega = 1$. Normalize $\omega$ as $\frac{\omega}{\omega_p}$

$$\epsilon \rightarrow \text{ripple width}$$

$$\Rightarrow |H(j\omega)|^2 = \frac{1}{1 + \left( \frac{\omega}{\omega_B} \right)^{2n}} = \frac{1}{1 + \epsilon^2 \left( \frac{\omega}{\omega_B} \right)^2} \quad \left\langle \text{renormalized} \right\rangle$$

at $\omega = 1$

$$|H(j\omega)|^2 = \frac{1}{1 + \left( \frac{\omega}{\omega_B} \right)^{2n}} = \frac{1}{1 + \epsilon^2} \Rightarrow \omega_B = \frac{\epsilon}{\epsilon^{1/n}}$$

$$\Rightarrow \text{At the passband corner}$$

$$\alpha_{\text{max}} = 10 \log \left( 1 + \epsilon^2 \right)$$

$$\Rightarrow \epsilon^2 = 10^{-0.1 \alpha_{\text{max}}} - 1 \quad \left\langle \text{eqn} \right\rangle$$
At the passband:

\[ d_{\text{min}} = 10 \log (1 + 10^{2n} w_s^2) \rightarrow 3 \]

From (3) and (2):

\[ d_{\text{min}} = 10 \log \left[ 1 + \left( 10^{0.1n_{\text{max}}} - 1 \right) w_s^2 \right] \]

\[ w_s^2 = \frac{10^{0.1n_{\text{min}}} - 1}{10^{0.1n_{\text{max}}} - 1} \rightarrow 4 \]

\[ n = \log \left[ \frac{10^{0.1n_{\text{min}}} - 1}{10^{0.1n_{\text{max}}} - 1} \right] \]

\[ \text{2} \log w_s \]

\[ \Rightarrow \text{gives the order.} \]

Design procedure:

1. Using (3) set \( n \) to determine parameters and \( \Delta \) sets \( \Delta_{\text{max}} \)
2. Using (4) calculate the degree \( n \) and round it to the next largest integer
3. Calculate the normalizing frequency \( \omega_B = 2 \pi \Delta \)

3-dB Bandwidth

Matlab functions:

\texttt{buttord} \n\texttt{butter}
Oversampling:

\[ f_{in} = \frac{f_s}{1.024} \]

Noise now confined to the bandwidth of the LPF.

See Matlab file: Oversampling.m

\[ \text{Bandwidth} = \frac{f_s}{2.0SR} \]

oversampling \( \Rightarrow \) SNR has increased by a factor close to the oversampling ratio, OSR.

\[ N_{inc} = 0.5 \log_2 \text{OSR} \]

\( \frac{1}{2} \) bit increase per doubling in OSR.
$f = 20SR/f_a$

No bit resolution

Rate = $2f_a$

N + $\frac{1}{2} \log_2 OSR$

resolution.

$$SNR = 10 \log \left( \frac{A/2}{L \cdot OSR} \right) = \frac{(2^{N-1} \cdot D)^2}{2 \cdot L \cdot OSR}$$

$$= 10 \log \left( \frac{1}{2 \cdot 2^N \cdot OSR} \right)$$

$$SNR = 6.02 N + 1.76 + 10 \log_{10} OSR$$

End to end system looks like a Nyquist rate ADC with effective resolution $N + \frac{1}{2} \log_2 OSR$.

Lost only 1 bit quantizer (cheap quantizer)

Trading analog complexity with digital complexity.

Digital Decimation filter is automatically synthesized

Veriﬁed coding

More effort on optimizing analog front end of the circuit.

Can we do better than $\frac{1}{2}$ bit per doubling in OSR?
Use feedback to reduce error $e(t)$?

Reduce $|u - v|$, using high loop gain?

Consider

$$
(u(z) - z^2v(z))A + E(z) = V(z)
$$

$$
A\cdot u(z) + E(z) = V(z)(1 + A \cdot z^2)
$$

$$
\Rightarrow V(z) = \left(\frac{A}{1 + A \cdot z^2}\right) X(z) + \frac{E(z)}{1 + A \cdot z^2}
$$

$\Rightarrow$ system pole at $-A \to -\infty$

$\Rightarrow$ system is not stable at all!

Error $|u - v| \to \infty \text{ as } A \to \infty$

Error $|u - v|$ blows up.

Ex.: $V[0] = v$

$\Rightarrow$ $|u - v| = 1 \Rightarrow y = -100$

$\Rightarrow$ $|u - v| = 100 \Rightarrow y = +10$

The geometric series explodes.

Doesn't work at all!
Too much delay in the loop causes instability.

- A constant large gain doesn't work.
  - Need to find another way to stabilize the loop.
  - \( u(t) \) has low frequency content compared to \( f_a \).
  - Apply high gain at low frequencies to reduce quantization noise.
  - At high frequencies, keep the gain low to stabilize the loop.

Use frequency-dependent gain.

Replace \( A \) by \( L(z) \) to loop fit.

\[ \text{Example} \Rightarrow \text{1st order}: \quad L(z) = \frac{1}{1-z^{-1}} \]

\[
\begin{align*}
& \begin{bmatrix} u(z) - z^{-1} v(z) \end{bmatrix} L(z) + E(z) = V(z) \\
\Rightarrow & \quad u(z) L(z) + E(z) = \frac{V(z)}{1+z^{-1}} \\
\Rightarrow & \quad V(z) = E(z) \left( \frac{1}{1+z^{-1}} \right) + \frac{L(z)}{1+z^{-1}} u(z)
\end{align*}
\]

\[ \text{STF}(z) = \frac{\frac{1}{1-z^{-1}}}{1 + \frac{z^{-1}}{1-z^{-1}}} = 1 \]

\[ \text{NTF}(z) = \frac{1}{1 + \frac{z^{-1}}{1-z^{-1}}} = (1-z^{-1}) \rightarrow \text{High pass response} \]
Oversampling

Oversampling: Time-domain waveforms

File: oversampling1.m

© Vishal Saxena
Oversampling

\[
\sigma_e^2 = 0.30483
\]

\[
\sigma_q^2 = 0.0082827
\]

File: oversampling1.m