Spectral Estimation Basics

Variants of Fourier transform

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**DTFT (Discrete Time Fourier Transform)**

\[X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}\]

- \(x[n]\) is absolutely summable or
  - \(\sum_{n=-\infty}^{\infty} |x[n]| < \infty\)
  - \(\sum_{n=-\infty}^{\infty} (x[n])^2 < \infty\)

\(\Rightarrow x[n]\) cannot be periodic

\(x(e^{j\Omega})\) is periodic with period \(2\pi\)

- \(2\pi\) corresponds to \(\Omega = 2\pi f_s\)

\(\Rightarrow\) similar to the sampled spectrum

**Example:**

\[x[n] = \begin{array}{c}
\begin{array}{cccccccc}
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\end{array}\]

\[x(e^{j\Omega}) = \sum_{n=0}^{4} x[n] e^{-j\Omega n}\]

\[= \frac{e^{-j\frac{\pi}{4}} (e^{j\frac{4\pi}{4}} - e^{-j\frac{4\pi}{4}})}{e^{-j\frac{\pi}{4}} (e^{j\frac{4\pi}{4}} - e^{-j\frac{4\pi}{4}})}\]

\[= e^{-j\frac{\pi}{4}} \frac{\sin(\frac{\pi}{2})}{\sin(\frac{\pi}{2})} \]

average delay = 2
Relation with $Z$-transform:

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \]

\[ x(e^{j\omega}) = x(z) \bigg|_{z = e^{j\omega}} \]

\[ \Rightarrow x(z) \text{ evaluated along the unit circle.} \]

But DTFT is continuous in frequency,

\( \Rightarrow \) not good for computations (using digital computers).

\( \Rightarrow \) need some transform which is also discretized in frequency axis. (Fourier series is discrete in frequency!)

Consider a finite length sequence, \( x[n] \) of length \( N \).

\( x[n] = 0 \) outside \( 0 \leq n \leq N-1 \)

Now create a periodic sequence \( \tilde{x}[n] \):

\[ \tilde{x}[n] = \sum_{\nu=-\infty}^{\nu} x[n - \nu N] \]

\[ \text{or } \tilde{x}[n] = x[n \text{ modulo } N] = x\left[(n)_N \right]. \]

Thus \( x[n] = \left\{ \begin{array}{ll} \tilde{x}[n], & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{array} \right. \)
Since $x[n]$ is periodic with period $N$, $X[k]$ can be represented as a summation of complex exponentials with a frequency equal to the integer multiples of the fundamental frequency $(2\pi/N)$.

**Periodic complex exponentials**

$$e_{k}[n] = e^{j(\frac{2\pi}{N})kn} = e_{k}[n+nN].$$

Then

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j(\frac{2\pi}{N})kn} \quad \text{Discrete Fourier Series representation (DFS)}$$

think of

$$x(t) = \sum_{k=-\infty}^{\infty} A_k e^{j2\pi ft}$$

with additional factor $N \rightarrow \text{period}$.

Test: $E_{k+lN}[n] = e^{j(\frac{2\pi}{N})((k+lN)n)}$

$$= e^{j\frac{2\pi}{N}(kn + lNn)}$$

$$= e^{j\frac{2\pi}{N}kn} \cdot e^{j\frac{2\pi}{N}lNn}$$

$$= e^{j\frac{2\pi}{N}kn} = e_{k} \quad \text{periodic with } N$$

Thus we need only $e_{k}[n]$ to $e_{k+N}[n]$ to represent $x[n]$.

$$\Rightarrow \quad X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j(\frac{2\pi}{N})kn} \quad \text{DFS coefficient}$$

only $N$ frequency components

where

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(\frac{2\pi}{N})kn}$$

think of

$$a_n = \frac{1}{N} \int_{x} x(t)e^{-j2\pi ft} dt$$

Since $x[n]$ and $e^{-j(\frac{2\pi}{N})kn}$ are both periodic with $N$

$$\Rightarrow X[k] \text{ is also periodic with period } N.$$
for convenience we use \( W_N = e^{-j\frac{2\pi}{N}} \)

$$
\text{Analysis Eq: } \quad \hat{x}[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] W_N^{-nk} \\
\text{Synthesis Eq: } \quad x[n] = \sum_{k=0}^{N-1} \hat{x}[k] W_N^{nk}
$$

both \( x[n] \) and \( \hat{x}[k] \) are periodic with \( N \).

* for properties of DFT, refer to Oppenheim & Schafer, DSP book.

Now, define \( X[k] = \{ \hat{x}[k], 0 \leq k \leq N-1 \} \) \( \hat{x} \) are the frequency
domain indices

- \( X[k] \) is an \( N \)-point sequence
- \( X[k] \) is the Discrete-Fourier Transform (DFT)

Also, we can show that \( \hat{x}[k] = X(e^{j\omega}) \) \( \omega = \frac{2\pi}{N} \)

\( X[k] \) for \( k \epsilon [0, N-1] \)

DFT is DTFT sampled in frequency domain as \( \omega = \frac{2\pi k}{N} \).
DFT computation

1. Make periodic extensions of \( x[n] \) to obtain \( x'[n] \) with period \( N \).
2. Find DFT coefficients \( x[k] \) of \( x'[n] \).

\[
\text{DFT } x[k] = \begin{cases} 
  x[k], & 0 \leq k \leq N-1 \\
  0, & \text{otherwise}
\end{cases}
\]

\[
\rightarrow \text{N-point FFT, usually } N = 2^L.
\]

Example:

\[ x[n] \]

\[ \begin{array}{cccccc}
  & & \text{I} & \text{I} & \text{I} & \text{I} \\
0 & 1 & 2 & 3 & 4 & 5
\end{array} \]

\[ N=5 \]

we had:

\[
x(e^{j\omega}) = e^{-j\omega} \frac{\sin \left( \frac{\omega}{2} \right)}{\sin \left( \frac{\omega}{4} \right)}
\]

Now:

\[
x[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} = \sum_{n=0}^{N-1} 4 \left( e^{-j\frac{2\pi}{5}} \right)^n = 4 \left( e^{-j\frac{2\pi}{5}} \right) \frac{1 - e^{-j\frac{2\pi kn}{N}}}{1 - e^{-j\frac{2\pi}{5}}}
\]

\[
= \frac{e^{-j\frac{2\pi kn}{N}} \left( e^{j\frac{2\pi kn}{N}} - e^{-j\frac{2\pi kn}{N}} \right)}{e^{-j\frac{2\pi kn}{N}} \left( e^{j\frac{2\pi kn}{N}} - e^{-j\frac{2\pi kn}{N}} \right)} = e^{-j\frac{2\pi kn}{N}} \frac{\sin \left( \frac{k\pi n}{5} \right)}{\sin \left( \frac{k\pi}{10} \right)}
\]

observe that \( x[k] \approx x(e^{j\omega}) \bigg|_{\omega = \frac{2\pi kn}{N}} \)

\[ \text{DTFT } x(e^{j\omega}) \]

\[ \text{DFT } x[k] \]

DFT is a discrete representation of \( x(e^{j\omega}) \) sampled at \( \omega = \frac{2\pi nk}{N} \)
Signal estimation using DSP/Matlab.

Example: characterizing the distortion of a s/H using a single tone input.

\[ \text{Vin} \xrightarrow{\text{Non-linear System}} \text{V}_{\text{out}} \xrightarrow{\text{contains harmonics}} \text{Ideal s/H} \]

Use a discrete time periodic sequence with period N.

\[ V[n] = V[n+N] \]

Then \( V[n] \) can be represented as DFS.

\[
V[n] = \sum_{k=0}^{N-1} V[k] e^{j2\pi kn/N} \rightarrow \text{complex DFS coefficients } V[k] = |V[k]| e^{j\phi[k]}
\]

\( V[k] \) are easily computed using FFT.

- \( N \rightarrow \text{record length} = \text{FFT size} \)
- \( \{0, \frac{2\pi}{N}, \frac{2\pi}{N} \cdot 2, \ldots, \frac{2\pi}{N} \cdot (N-1)\} \)

\( \frac{N}{f_s} \rightarrow \text{time in continuous-time axis} \)

\( \frac{f_s}{N} \rightarrow \text{resolution of the FFT} \)

\( \frac{f_s}{N} \rightarrow \text{spacing between the tones} \)

\[
\frac{1}{2} \xrightarrow{\frac{f_s}{N}} \frac{1}{2^{N-1}} \leq \text{resolution or bin size}
\]

P.S.

- \( M \rightarrow \text{record length} = \text{size of the data collected from simulation or measurement} \)
- \( \frac{N}{M} \rightarrow \text{resolution or bin size} \)
- \( \text{If the FFT is taken over the whole record length then} \)
  \( \text{FFT size} = M, \text{i.e. } N=M \).