Distributed feedback path around the quantizer:

\[ \text{L'(s)} = \frac{K_0 + \frac{K_1}{s} + \frac{K_2}{s^2}}{s} \]

We had \[ L(z) = \frac{-2z+1}{(z-1)^2} \] Extra feedback path provides the extra control parameters in the loop response.

Now if when the ELT is completely compensated:

\[ R_0 Z^{-1} + k_1 (\text{RHS of } A) + k_2 (\text{RHS of } B) = \frac{(2z+1)}{(z-1)^2} = \frac{2z-1}{(z-1)^2} \]

Going through the algebra we get:

\[ 0.5z^2k_2 - 2k_1 + k_0 = 0 \]
\[ 0.5z^2 - 2 + 0.5z^2 \]
\[ (6.5z^2 + 1-2)k_1 + k_0 = 2 \]
\[ -(0.5z^2 - 2)k_2 + (1-2z)k_1 + 2k_0 = 1 \]

Solving this set of equations we get:

\[ \{k_0, k_1, k_2\} = \{0.5z^2 + 0.5z^2, 1.5 + 2, 2\} \]

Verify for \( z = 0 \), \( \{k_0, k_1, k_2\} = \{0, 1.5, 2\} \) same as \( x_k \).

\( \Rightarrow k_1 \) is turned on and \( k_0 \) is added.

Lo this process requires one to go back and forth the \( s \) and \( z \) domain.

Ladder algebra is powerful for higher order modulators.
Pavan's solution for loop filter comprising of ideal integrators (no resonators).

The technique:

If the CT ΔΣ loop filter is

\[ L(s) = \frac{k_1}{s} + \frac{k_2}{s^2} + \cdots + \frac{k_n}{s^n} \]

with no external delay.

What value of coefficients \( \{k_1, k_2, \ldots, k_n\} \) must be chosen so that the NTI remains same even when there is an ELD of \( T \).

Conceptually we can compensate for ELD by cascading \( L(s) \) with a block with \( TF \equiv e^{st} \).

![Block Diagram](image)

Clearly, \( e^{st} e^{-st} = e^{2st} \) \((t + T)^{-1} \) is non-causal.

It is not realizable in practice.

We get an interesting case when the input is piecewise constant and we are only interested in the sampled output. \( L \) can be reinterpreted to obtain

\[ \hat{F}(s) = H(s) e^{st} = \frac{k_1 e^{st}}{s} + \frac{k_2 e^{st}}{s^2} + \cdots + \frac{k_n e^{st}}{s^n} \]
A. NRZ DAC

For NRZ DAC, expand $e^{st}$ as the polynomial is $s^i$, such that

\[ \frac{e^{st}}{s^i} \text{ is truncated beyond the } i^{th} \text{ power.} \]

\[ \frac{1}{s^i} (1 + s^2 t + \frac{s^{2i} t^2}{2}) = \frac{1}{s^i} + \frac{t}{s} + \frac{t^2}{s^2} \rightarrow (2) \]

\[ \frac{e^{st}}{s^i} \rightarrow \frac{1}{s^i} (1 + \cdots + \frac{s^i t^i}{i!}) = \frac{1}{s^i} + \frac{t}{s^{i-1}} + \frac{t^2}{s^2} \rightarrow (3) \]

Then, the loop filter $H(z)$, whose samples are identical to $L(z)$, are given by the weighted summation of the RHS of these equations by $k_1, \ldots, k_n$, respectively.

\[ L(z) = \frac{(k_1 z^2 + k_2 \frac{z^2}{2} + \cdots + k_n \frac{z^n}{n!})}{L} + \left( \frac{k_1 + k_2 z + \cdots + k_n \frac{z^{n-1}}{n-1}}{s} \right) + \cdots \]

\[ \Rightarrow \quad k_0' = k_1 z + k_2 \frac{z^2}{2} + \cdots + k_n \frac{z^n}{n!} = \sum_{i=1}^{n} k_i \frac{z^i}{i!} \]

\[ k_1' = k_1 + k_2 z + \cdots + k_n \frac{z^{n-1}}{n-1} = \sum_{i=1}^{n} k_i \frac{z^{i-1}}{i-1} \]

\[ k_n' = k_n \]

Verify the result for the second order case

by using $\{k_1, k_2\} = \{1, 5, 13\}$
Proof:

$u(t)$ — unit step function integrated $K$ times,

$u(t) = u(t)$ — unit step function

The DAC pulse is $u(t) - u(t-1)$, initially assume $z = 0$

$y_i(t)$ — output of the $i^{th}$ integrator.

No order of the loop

$y_i(t)$ — output

we have

$y_i(t) = \sum_{i=1}^{N} k_i x_i(t) = \sum_{i=1}^{N} k_i (u_i(t) - u_i(t-1))$

the sampled output

$y_i[n] = \sum_{i=1}^{N} k_i x_i[n] = \sum_{i=1}^{N} k_i (u_i[n] - u_i[n-1])$

when the DAC pulse is delayed by $\tau$, the integrator and loop filter output become $x_i(t-\tau)$ and $y_i(t-\tau)$.

Using Taylor series for $0 < \tau < 1$, we have the ideal sampled output of the $i^{th}$ integrator can be expressed as:

$x_i[n] = x_i(t)\bigg|_{t=n} = x_i(t-\tau)\bigg|_{t=n} + \frac{dx_i(t-\tau)}{dt}\bigg|_{t=n} \cdot \tau + \frac{\tau}{2!} \frac{d^2 x_i(t-\tau)}{dt^2}\bigg|_{t=n} \cdot \frac{\tau^2}{2} + \cdots + \frac{\tau^n}{n!} \frac{d^n x_i(t-\tau)}{dt^n}\bigg|_{t=n} \cdot \frac{\tau^n}{n!} + \cdots$

Now, since

$\left. \frac{dx_i(t-\tau)}{dt} \right|_{t=n} = x_i(t-\tau) \frac{dx_i(t-\tau)}{dt} \bigg|_{t=n} = 0$

(1) reduces to

$x_i[n] = \left[ x_i(t-\tau) + \frac{\tau}{2!} x_i(t-\tau) + \frac{\tau^2}{3!} x_i(t-\tau) + \cdots + \frac{\tau^n}{n!} x_i(t-\tau) \right] \bigg|_{t=n}$
\[
\frac{f(t)}{L} = \frac{f(0)}{L} + \frac{f(t_0)}{L} \frac{(t-t_0)}{L} + \frac{f''(t)}{L} \frac{(t-t_0)^2}{L^2} + \cdots + \frac{f^{(n)}(t)}{L} \frac{(t-t_0)^n}{L^n} + \cdots
\]

\[
f(t) = x_i(t)
\]

\[
t_0 = \frac{t-2}{n}
\]

\[
x_i(t) = x_i(t-2) + \frac{x_i'(t-2)}{2} + \cdots + \frac{x_i^{(n)}(t-2)}{n!}
\]

\[
x_i(t) \bigg|_{t=n} = x_i(t-2) \bigg|_{t=n} + \frac{x_i'(t-2)}{2} \bigg|_{t=n} + \cdots + \frac{x_i^{(n)}(t-2)}{n!} \bigg|_{t=n}
\]

\[
x_i(t-2) = x_{i-1}
\]

\[
x_0'(t-2) = 0
\]

\[
x_i[n] = \left[ x_i(t-2) + \frac{x_i'(t-2)}{2} + \cdots + \frac{x_i^{(n)}(t-2)}{n!} \right] \bigg|_{t=n}
\]

\[
\frac{1}{s^n} + \frac{2}{s^{n-1}} + \cdots + \frac{n!}{s} = \frac{1}{s^n} \left[ e^{st} \right]_{t=0}^{t=\text{final}}
\]
\[ \hat{H}(s) = L(s) e^{ST} = k_1 \frac{e^{ST}}{s} + k_2 \frac{e^{2T}}{s^2} + \cdots + k_n \frac{e^{nT}}{s^n}. \]

\[ x_0(t) = u(t) - u(t-1) \]

\[ x_1(t) \]

\[ x_0(n) = x_0(n-2) + 2x_0(n-2) \]

\[ x_1(n) = x_1(n-2) + 2x_1(n-2) \]

\[ x_2[n] \text{ can be determined from } x_2[n-2], x_1[n-2] \text{ and } x_0[n-2]. \]

\[ x_2(n) \]

\[ x_2(n) \text{ can be linearly combined to estimate green dots.} \]
1) Even with delayed DAC pulse, the ideal output samples of the $i^{th}$ integrator can be generated by combining the output samples of the $i^{th}$ integrator and the preceding $(i-1)$ integrators as well as the input to the loop filter $x(t-2)$.

2) In frequency domain, we can say that the output samples of the $i^{th}$ integrator, can be obtained by sampling the output of a filter whose transfer function is given by

$$\frac{1}{s^i} + \frac{s}{s^i} + \cdots + \frac{1}{L_i} s \frac{s}{s^i} = \frac{1}{s^i} e^{sL_i}$$

3) TF of the compensated loop filter is given by

$$\hat{L}(s) = \frac{k_{o1}}{s^m} \left(1 + sZ + \frac{s^2Z^2}{L_1} + \cdots + \frac{s^mZ^m}{L_m}\right)$$

$$\cdots + \frac{k_{oi}}{s^i} \left(1 + sZ + \frac{s^2Z^2}{L_1} + \cdots + \frac{s^mZ^m}{L_m}\right)$$

$$\cdots + \frac{k_{o1}}{s} (1 + sZ)$$

4) From 3) the direct path comes out as a direct consequence

$$k_{o1}' = k_{i1}Z + k_{i2} \frac{Z^2}{L_1} + \cdots + k_{io} \frac{Z^{io}}{L_m}$$
(i) \( z < 0.5 \), the order of the system doesn't increase.

L-ELD compensation alone by careful tuning

same as NRZ derivation, but the expansion of \( e^{zT} \) in \( \frac{e^{zT}}{s^i} \) is

truncated after \((i-1)\)th power of \( s \).

\[
\frac{e^{zT}}{s^i} \rightarrow \frac{1}{s^i} \left( 1 + \frac{(zT)^{i-1}}{(i-1)!} \right) = \frac{1}{s^i} + \frac{z}{s^{i-1}} + \frac{z^{i-1}}{s^{i-1} \cdot i-1}
\]

\( z \) is the compensated loop \( TF \) is

\[
\gamma(s) = \frac{k_i + k_{i+1}z + \cdots + k_nz^{n-1}}{s^i} = \frac{k_i + k_{i+1}z + \cdots + k_nz^{n-1}}{s^i}
\]

(ii) \( z > 0.5 \), a direct path is necessary in addition to coefficient term.

Let see ref. for details.

3) Simple Randle-when all is s-domain!

Lo method requires that the higher order derivatives of the \( n^{th} \) integrator's output become 0 when driven by a piecewise constant Dirac pulse.

Lo true when NTF with complex zeros is used.

Lo false responses now contain sine and cosine.

Lo no easy solution exist "

Lo the low-pass formulas do an acceptable job of stabilizing the

loop-delay for large OSR, but-pass OSMs.

Lo No Solution \( \rightarrow \) BP-OSMs.
Issues with Table-based method:

- Certainly need mathematical analysis for better understanding of the system.
- The algebra (for the general case) is tedious and uncertain.
  - TF of real ramp have several poles/zeros due to finite Amp, etc. (assuming Atami-like implementation).
  - Obtaining their pole/zero locations not an easy task.

- The system may not have a solution when the integrators are non-ideal and the poles of $L(x)$ are different from the poles of the integrator paths.
  - E.g., for integrator with finite gain, poles of real log filter will not be at $\omega = 1$.
  - Can not be solved.
Numerical fitting approach

To implemented in the realize NTF \( z \) functions.

\[ \mathbf{e}[z] \rightarrow \text{column vectors of } \mathbf{N} \text{ samples} \]

\[ \text{eg for } \text{NTF}(z) = (1-z)^{-2}, \text{ we have} \]

\[ \mathbf{e}[z] = [0 \ 2 \ 3 \ \ldots \ ]^T. \]

The column vectors formed by \( \mathbf{N} \) samples of the filter responses of the direct path and the integrator outputs are denoted as

\[ \mathbf{e}_0[z] = [0 \ 1 \ 0 \ \ldots \ ]^T \]

\[ \mathbf{e}_1[z] = [0 \ (1-z) \ 1 \ \ldots \ ]^T \]

\[ \mathbf{e}_2[z] = [0 \ 0.5(1-z)^2 \ (1.5-z) \ \ldots \ (N-0.5-z)]^T \]

Choose \( \mathbf{n} \) such that it is much longer than the number of unknowns to be determined. Then we have the weighting coefficient \( \mathbf{K} = [-k_0, k_1, k_2]^T \) determined by solving

\[ \begin{bmatrix} \mathbf{e}_0[z] & \mathbf{e}_1[z] & \mathbf{e}_2[z] \end{bmatrix} \mathbf{K} = \mathbf{h}[z]. \]

\[
\begin{bmatrix}
1 & 1-z & 0.5(1-z)^2 \\
0 & 1 & 1.5-z \\
0 & 1 & N-0.5-z
\end{bmatrix}
\begin{bmatrix}
k_0 \\
k_1 \\
k_2
\end{bmatrix}
=
\begin{bmatrix}
d_0 \\
d_1 \\
d_2
\end{bmatrix}
\]

More equations than unknowns, with ideal integrators, the above set of equations admit a unique solution. \( \Rightarrow \) \( \mathbf{K} \) is independent of \( \mathbf{N} \).

Let\( \mathbf{d}_0 \) boost away with tedious algebra. Heavily obtained from simulation and

(1) Other issues when real opamps are used
(2) Read paper by Shanthi.